Weighted Moments for a Branching Process in a Random Environment

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1. GW process

Z_n is called a branching process if

$$Z_0 = 1 \quad \text{and} \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \;\text{ for }\; n \geq 0, \tag{1}$$

where $X_{n,i}$ is the number of offspring of the ith particle in the n generation, iid, $X_{n,i} \sim (p_j)$.

For a Galton-Watson process (Z_n). Let $m = \mathbb{E}Z_1$. Let q be the extinction probability.

$$\begin{aligned} m &< 1 \text{ is subcritical}, q = 1; \\ m &= 1 \text{ is critical}, q = 1; \\ m &> 1 \ \text{ is supercritical}, q < 1. \end{aligned}$$

When $m > 1$, $W = \lim_{n\to\infty} \frac{Z_n}{m^n}$ is non-degenerate if and only if $\mathbb{E}Z_1 \log Z_1 < \infty$.

Under this condition, the moments of the limit variable W has been studied by many authors.

Of particular interest is the existence of the weighted moments of W of the form $\mathbb{E}W^\alpha l(W)$ where $\alpha > 1$ and l is a positive function slowly varying at ∞.

(1) Bingham and Doney 1974 showed that $\mathbb{E}W^\alpha l(W) < \infty$ if and only if $\mathbb{E}Z_1^\alpha l(Z_1) < \infty$, when $\alpha > 1$ is not an integer.

(2) Alsmeyer and Rösler 2004 proved that the same result remains true for all non-dyadic integer $\alpha > 1$ (not of the form 2^k for some integer $k \geq 0$).

(3) Liang and Liu 2013a proved that the result holds true for *all* $\alpha > 1$.

2. GW process in a random environment

A branching process in a random environment is a natural and important extension of the GW process.

Let $\xi = (\xi_n)_{n\geq 0}$ be i.i.d. and $p(\xi_n)$ be a sequence of probability distribution.

$(Z_n)_{n\geq 0}$ is called a branching process in the random environment: if (1) is satisfied and $X_{n,i}$ is a sequence of conditional independent and distribution random variables, where $X_{n,i} \sim (p_j(\xi_n))$.

Let $m_n = \mathbb{E}_\xi X_{n,i}$.

$$\begin{aligned}
&\mathbb{E}\log m_0 < 0 \text{ is subcritical}, q = 1;\\
&\mathbb{E}\log m_0 = 0 \text{ is critical}, q = 1;\\
&\mathbb{E}\log m_0 > 0 \text{ is supercritical}, q < 1.
\end{aligned}$$

Let

$$W_n = \frac{Z_n}{\Pi_n}, \quad n \geq 0$$

where $\Pi_n = \mathbb{E}_\xi Z_n$ and the limit be W.

We consider the supercritical case. W is non-degenerate (which is also equivalent to the convergence in L^1 of (W_n)) if and only if

$$\mathbb{E}\left(\frac{Z_1 \log^+ Z_1}{m_0}\right) < \infty \tag{2}$$

(see Athreya and Karlin (1971b) for the sufficiency and Tanny (1988) for the necessity).

$\star$ The probability $\mathbb{P}$ is usually called *annealed law*, while $\mathbb{P}_\xi$ is called *quenched law*.

3. Weighted moments of W

We will consider the existence of weighted moments of W of the form $\mathbb{E}_\xi W^\alpha l(W)$ and $\mathbb{E} W^\alpha l(W)$, for which we will show that the existence conditions are quite different between the annealed case and the quenched case.

Meanwhile we also consider the same problem for the maximum variable

$$W^* = \sup_{n \geq 1} W_n.$$

Definition 1

Let l be a positive measurable function, defined on some neighborhood $[X, \infty)$ of infinity, and satisfying

$$l(\lambda x)/l(x) \to 1 \ (x \to \infty) \ \ \forall \lambda > 0;$$

then l is said to be **slowly varying** (in Karamata's sense).

Theorem 1 (Bingham, Goldie and Teugels, 1987, Theorem 1.3.1)

Any slowly varying function l slowly varying at ∞ is of the form

$$l(x) = c(x) \exp\left(\int_{a_0}^{x} \frac{\varepsilon(t)}{t} dt \right), \ \ x > a_0,$$

where $a_0 > 0$, $c(\cdot)$ and $\varepsilon(\cdot)$ are measurable with $c(x) \to c$ for some constant $c \in (0, \infty)$ and $\varepsilon(x) \to 0$ as $x \to \infty$.

Theorem 2 is the necessary and sufficient condition for the existence of the annealed weighted moments of W.

Theorem 2 (Liang and Liu 2013)

(Annealed case) Let $\alpha > 1$ *and* $l : [0, \infty) \to [0, \infty)$ *be a function slowly varying at* ∞. *Assume that* $\mathbb{E} m_0^{1-\alpha} < 1$. *Then the following assertions are equivalent:*

(1) $\mathbb{E} W_1^{\alpha} l(W_1) < \infty$;

(2) $\mathbb{E} W^{*\alpha} l(W^*) < \infty$;

(3) $0 < \mathbb{E} W^{\alpha} l(W) < \infty$.

We obtain the necessary and sufficient conditions for the existence of quenched weighted moments of W.

Theorem 3

Let $l(x)$ be a function slowly varying at ∞ and $\phi(x) = x^\alpha l(x)$ with $\alpha > 1$. Assume $\mathbb{E}\left(\frac{Z_1 \log^+ Z_1}{m_0}\right) < \infty$ and $\mathbb{E} \log m_0 < \infty$. Then the following assertions are equivalent:
(1) $\mathbb{E} \log^+ \mathbb{E}_\xi \phi(W_1) < \infty$; *(2)* $\mathbb{E}_\xi \phi(W) < \infty$; *(3)* $\mathbb{E}_\xi \phi(W^*) < \infty$.

Remark 1

The results extend a theorem by Huang and Liu 2014 about the special case where l is a constant.
The general case where l is not necessarily a constant makes the proof much more difficult. And improve the sufficient condition given by Li, Hu and Liu in 2011 where a completely different method was used.

Remark 2

For the equivalence between (2) and (3) we do not need the condition $\mathbb{E}\log m_0 < \infty$. *Actually, this equivalence is a general result for martingales; we will prove it by establishing an extended version of Doob's inequality about weighted moments for nonnegative submartingales, which is of independent interest; see Theorem 4 below.*

Theorem four is an extended Doob' s inequality for ϕ-moments on sub-martingale.

Theorem 4

Let $(f_n, \mathcal{G}_n)$ be a nonnegative submartingale convergent a.s. and in L^1. Let $\phi(x) = x^\alpha l(x)$, where $\alpha > 1$, l is a positive function slowly varying at ∞ and locally bounded on $[0, \infty)$. Then there exist two constants $C_0 > 0$ and $C_1 > 0$ depending only on ϕ, such that

$$\mathbb{E}\phi(f) \leq \mathbb{E}\phi(f^*) \leq C_0 + C_1 \mathbb{E}\phi(f),$$

where $f = \lim_{n\to\infty} f_n$ and $f^* = \sup_{n\geq 0} |f_n|$.

The difference with our results is that ϕ in Lemma one is a convex function.

Lemma 1 (Alsmeyer and Rösler 2006, Proposition 1.1)

Let $\phi : [0, \infty) \to [0, \infty)$ be an unbounded, nondecreasing convex function, with $\phi(0) = 0$,

$$p_\phi := \inf_{0<x<\infty} \frac{x\phi'(x)}{\phi(x)} > 1 \quad \textit{and} \quad p_\phi^* := \sup_{0<x<\infty} \frac{x\phi'(x)}{\phi(x)} < \infty,$$

where $\phi'(x)$ denotes the right derivative of ϕ at x. Then for each $n \geq 0$, the maximum variable $f_n^* = \sup_{0\leq k\leq n} f_k$ satisfies

$$\mathbb{E}\phi(f_n^*) \leq \left(\frac{p_\phi}{p_\phi - 1}\right)^{p_\phi^*} \mathbb{E}\phi(f_n).$$

Proof of Theorem 4. Step 1: Let ϕ_1 be defined and $\alpha = \alpha_0 + b$ where $b > 0$ and $\alpha_0 > 1$. Let $\delta > 0$ be small enough such that $\alpha - \delta > \alpha_0$ and a_1 be large enough such that

$$\alpha - \delta < \frac{\alpha\phi(x)}{\phi_1(x)} < \alpha + \delta \qquad \text{for all } x \geq a_1.$$

Step 2: We set $\phi_2(x) = \phi_1(x)$ if $x \geq a_1$; $\phi_2(x) = x^{\alpha_0} a_1^b l_1(a_1)$ if $x \in [0, a_1)$. ϕ_2 is a convex function.

Step 3:By calculation, we know $p_{\phi_2} > 1$ and $p^*_{\phi_2} < \infty$. Then, by Lemma 1, we have

$$\mathbb{E}\phi_2(f_n^*) \leq (\frac{p_{\phi_2}}{p_{\phi_2} - 1})^{p^*_{\phi_2}} \mathbb{E}\phi_2(f_n), \quad \text{where} \quad f_n^* = \sup_{0 \leq k \leq n} f_k.$$

Step 4:By Jensen's inequality and the monotone convergence theorem, we get the result of Lemma one with ϕ_2:

$$\mathbb{E}\phi_2(f^*) \le (\frac{p_{\phi_2}}{p_{\phi_2} - 1})^{p^*_{\phi_2}} \mathbb{E}\phi_2(f).$$

Step 5:Because $\phi_2(x) \sim \phi(x)$, as $x \to \infty$, we know that there exist two constants $C_0 > 0$ and $C_1 > 0$ depending only on ϕ, such that

$$\mathbb{E}\phi(f^*) \le C_0 + C_1 \mathbb{E}\phi(f).$$

4. Proof of Theorem 4

For a martingale sequence $\{(f_n, \mathcal{G}_n) : n \geq 1\}$, $d_n = f_n - f_{n-1}$, $f^* = \sup_{n\geq 1} |f_n|$ and $d^* = \sup_{n\geq 1} |d_n|$.

Lemma 2 (Burkholder-Davis-Gundy inequality, Chow and Teicher 1995, Theorem 2)

Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing and continuous function with $\Phi(0) = 0$ and $\Phi(2\lambda) \leq c\Phi(\lambda)$ for some $c \in (0,\infty)$ and all $\lambda > 0$.

(1) *For every $\beta \in (1,2]$, there exists a constant $B = B_{c,\beta} > 0$ depending only on c and β such that*

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(s(\beta)) + B\mathbb{E}\Phi(d^*); \tag{3}$$

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(s(\beta)) + B\sum_{n=1}^{\infty} \mathbb{E}\Phi(|d_n|),$$

with $s(\beta) = \left(\sum_{n=1}^{\infty} \mathbb{E}\left(|d_n|^{\beta} | \mathcal{G}_{n-1}\right) \right)^{1/\beta}.$

(2) If Φ is convex on $[0,\infty)$, then there exist constants $A = A_c > 0$ and $B = B_c > 0$, depending only on c such that

$$A\mathbb{E}\Phi(S) \leq \mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(S),$$

where $S = (\sum_{n=1}^{\infty} d_n^2)^{1/2}$. Moreover, for any $\beta \in (0,2]$,

$$\mathbb{E}\Phi(f^*) \leq B\mathbb{E}\Phi(S(\beta)),$$

where $S(\beta) = (\sum_{n=1}^{\infty} |d_n|^{\beta})^{1/\beta}$. If, additionally, for some $\beta \in (0,2]$ the function $\Phi_{1/\beta}(x) = \Phi(x^{1/\beta})$ is subadditive on $[0,\infty)$, then

$$\mathbb{E}\Phi(f^*) \leq B \sum_{n=1}^{\infty} \mathbb{E}\Phi(|d_n|).$$

Lemma 2 (Bingham, Goldie and Teugels, 1987, Theorem 1.5.6)

(Potter's Theorem) (1) If l is slowly varying then for any chosen constants $A > 1, \delta > 0$, there exists $X = X(A, \delta)$ such that

$$l(y)/l(x) \leq A \max\{(y/x)^{\delta}, (y/x)^{-\delta}\} \quad (x \geq X, y \geq Y).$$

(2) If, further, l is bounded away from 0 and ∞, on every compact subset of $[0, \infty)$, then for every $\delta > 0$ there exists $A' = A'(\delta) > 1$ such that

$$l(y)/l(x) \leq A' \max\{(y/x)^{\delta}, (y/x)^{-\delta}\} \;\; (x > 0, y > 0).$$

5. Reference

- G. Alsmeyer and U. Rösler. Maximal $\phi-$inequalities for nonnegative submartingales. Theory Probab. Appl. 50(1)(2006) 118-128.
- K. H. Bingham, C. M. Goldie, and J. L. Teugels. Regular Variation (Cambridge Univ: Press, Cambridge, 1987).
- Y. S. Chow and H. Teicher. Probability Theory: Independence, Interchangeability, Martingales (New York: Springer, 1995).
- Chunmao Huang and Quansheng Liu. Convergence in L^p and its exponential rate for a branching process in a random environment. Electron. J. Probab. 104(19) (2014) 1-22.
- Xingang Liang and Quansheng Liu. Weighted moments of the limit of a branching process in a random environment. Proceedings of the Steklov Institute of Mathematics. 282 (2013b) 127-145.

Thank you!